

A REMARK ON A CONSTRUCTION OF D.S. ASCHE

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ABSTRACT. It is shown that a classical construction of D.S. Asche of 72 equiangular lines in the 19 dimensional Euclidean space contains a subset of 54 equiangular lines embedded in an 18 dimensional subspace.

1. INTRODUCTION AND MAIN RESULTS

A set of n lines, spanned by the unit vectors v_1, \dots, v_n in a d -dimensional Euclidean space is called equiangular, if there is a common angle $\alpha \geq 0$, such that for all $i \neq j$, $i, j \in \{1, \dots, n\}$ we have $|\langle v_i, v_j \rangle| = \alpha$. The concept of equiangular lines was introduced in [4], and its theory was developed in [5], [6]. The maximum number of equiangular lines, denoted by $N(d)$, is known for $d \leq 13$, and in several sporadic dimensions, but for $d \geq 14$ the problem is wide open in general. For an up-to-date table with the most recent lower and upper bounds on $N(d)$, and for a list of additional references we refer the reader to [2], [3].

The purpose of this note is to establish the following result.

Theorem 1.1. *We have $N(18) \geq 54$.*

Proof. Consider the extended binary Golay code as described in [1, p. 719]. The codewords are represented by the vectors generated by the rows of the 12×24 matrix $\begin{bmatrix} I_6 & O & 0 & \mathbf{1} \\ O & I_6 & \mathbf{1}' & A \end{bmatrix}$, where O is the 6×6 zero matrix, $\mathbf{1}$ and its transpose $\mathbf{1}'$ are vectors of length 11 having all entries equal to 1, and A is a 11×11 circulant matrix with first row $[0, 1, 0, 0, 0, 1, 1, 1, 0, 1, 1]$. Let \mathcal{C} denote the 759-element subset of codewords of weight 8. Let e_i denote the standard basis vectors of \mathbb{R}^{24} for $i \in \{1, \dots, 24\}$, and let $e_\Sigma := \sum_{i=1}^{24} e_i$. For a codeword $x \in \mathcal{C}$ let $f(x) := (4x - 4e_1 - e_\Sigma)/\sqrt{80}$. Let $c_1 := e_2 + e_3 + e_{14} + e_{15} + e_{16} + e_{19} + e_{22} + e_{23} \in \mathcal{C}$, and let $c_2 := e_2 + e_3 + e_9 + e_{11} + e_{12} + e_{13} + e_{21} + e_{24} \in \mathcal{C}$, and finally, let $m := 2e_4 - e_5 - e_6 + 2e_7 - e_8 - e_{10} + 2e_{17} - e_{18} - e_{20} - 3e_{22} + 3e_{23} \in \mathbb{R}^{24}$. Then one readily verifies that the subset of all vectors $f(d)$, $d \in \mathcal{C}$, for which $\langle d, e_1 \rangle = 1$, and $\langle f(d), 4e_1 + e_\Sigma \rangle = \langle f(d), e_1 - e_2 \rangle = \langle f(d), e_1 - e_3 \rangle = \langle f(d), c_1 \rangle = \langle f(d), c_2 \rangle = \langle f(d), m \rangle = 0$ forms an equiangular line system of $54 = 3 \cdot 18$ lines in \mathbb{R}^{18} with common angle $1/5$. \square

The proof of Theorem 1.1 is based on a construction of D.S. Asche who exhibited 72 equiangular lines in \mathbb{R}^{19} . Indeed, the only contribution of this paper is presenting the vector $m(c_1, c_2)$ which then removes 18 out of these 72 lines. Asche's construction is described in D. Taylor's thesis [8, p. 124] (a more accessible reference is [3, Example 5.19]), who remarked that $N(18) \geq 48$ follows. In an independent construction S. Snover used the residual of a Steiner triple system on 19 symbols to show the lower bound $N(17) \geq 48$. See [5] regarding the historical remarks, and [3, Example 5.18] for an explicit construction.

Remark 1.1. We offer the following geometric interpretation of the construction described in the Proof of Theorem 1.1. From the 72 equiangular lines (represented by unit vectors) coming from Asche's construction, we remove two 9-subset of unit vectors

Date: March 14, 2017. Preprint. This research was supported in part by the Academy of Finland, Grant #289002.

which form two disjoint cliques aligned in the ‘right’ way. More precisely, we remove a subset of 18 vectors u_i and v_i , $i \in \{1, \dots, 9\}$ for which the following conditions hold: $\langle u_i, u_j \rangle = \langle v_i, v_j \rangle = 1/5$, $i \neq j$, $i, j \in \{1, \dots, 9\}$ and in addition for every vector u_i , $i \in \{1, \dots, 9\}$ there are exactly two distinct vectors $v_j \neq v_k$, $j, k \in \{1, \dots, 9\}$ such that $\langle u_i, v_j \rangle = \langle u_i, v_k \rangle = 1/5$ (and the other way around). We observe that the two cliques are on either side of the 1-codimensional hyperplane orthogonal to m , as $\langle u_i, m \rangle = 6/\sqrt{5}$ whereas $\langle v_i, m \rangle = -6/\sqrt{5}$ for every $i \in \{1, \dots, 9\}$.

We conclude this note with three further observations.

First, the configuration given in the Proof of Theorem 1.1 is not extendible to a larger set of equiangular lines in \mathbb{R}^{18} .

Secondly, the Seidel matrix S corresponding to this newly constructed equiangular line system (with $[S]_{i,j} := 5(\langle v_i, v_j \rangle - \delta_{ij})$, where $i, j \in \{1, \dots, 54\}$, see [7]) has spectrum

$$\Lambda(S) = \{[-5]^{36}, [7]^6, [11]^8, [13]^2, [12 - \sqrt{37}]^1, [12 + \sqrt{37}]^1\}$$

(exponents denote multiplicities), and automorphism group size of $|\text{Aut}(S)| = 216 = 4 \cdot 54$.

Finally, amongst those sub-Seidel matrices of S , which are of order $n \in \{50, \dots, 53\}$ there is a unique one (up to equivalence) with integral spectrum. The spectrum of this Seidel matrix T of order 52 reads: $\Lambda(T) = \{[-5]^{34}, [3]^1, [5]^1, [7]^6, [11]^7, [13]^2, [17]^1\}$.

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